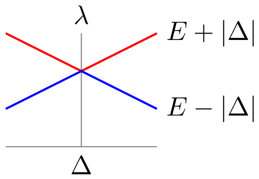
1. (a) All matrices A_j are Hermitian (in fact: symmetric), so they have a basis of eigenvectors with real eigenvalues. The matrices are translation invariant in the sense that $A_{mn} = f(m-n)$, i.e., the matrix elements depend only on the difference $m - n$.

   (b) See below for a possible set of eigenvectors.

   (c) $\lambda_1 = E+\Delta$, $\lambda_2 = E-\Delta$. There is a linear splitting of the eigenvalues.



   (d) True! $\mathrm{Tr}\, A_1 = 2E = \lambda_1 + \lambda_2$. $\mathrm{Tr}\, A_2 = 0$, $\mathrm{Tr}\, A_3 = 0$.

   (e) The matrix T applied to unit vector e_j will give unit vector e_{j-1}, so T translates unit vectors. The inverse of T moves the unit vector e_j to e_{j+1}.

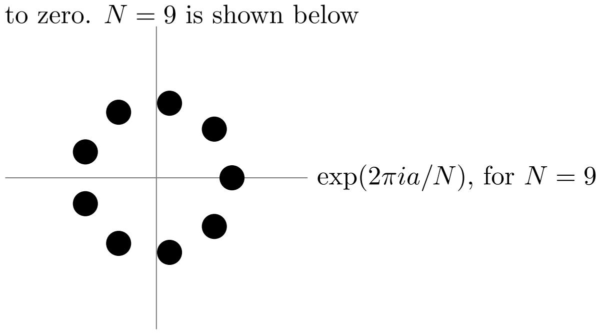
   (f) This follows from explicit matrix multiplication. If one is really smart, one could write $A_1 = \Delta(T + T^{-1}) + ET^2$, $A_2 = T + T^{-1}$ and $A_3 = 2(T + T^{-1}) + (T^2 + T^{-2})$, i.e. a power series expansion of A in T, and from there directly observe that A and T commute.

   (g) Orthonormal:

$$|v_k|^2 = \sum_{a=0}^{N-1} (v_a^k)^* v_a^k = \sum_{a=0}^{N-1} \frac{1}{N} = \frac{N}{N} = 1 \tag{1}$$

$$\langle v_k, v_k' \rangle = \sum_{a=0}^{N-1} (v_a^k)^* v_a^{k'} = \sum_{a=0}^{N-1} \frac{\exp(2\pi i a(k'-k))}{N} = \frac{0}{N} = 0 \tag{2}$$

   The second equation uses $k' - k \neq 0$, and in that case the corresponding exponents are uniformly distributed over the unit circle and sum to zero. $N = 9$ is shown below



   (h) $(Te_a)_m = \sum_n T_{mn}(e_a)_n = T_{ma} = \delta_{a-m,1}$, so $Te_a = e_{a-1}$.

(i) For clarity, I write the matrix T as $\hat{T}$ and the vector v as $\vec{v}$.

$$\hat{T}\vec{v}^k = \sum_a \hat{T}\frac{\exp(2\pi iak/N)}{\sqrt{N}}\vec{e}_a = \sum_a \frac{\exp(2\pi iak/N)}{\sqrt{N}}\hat{T}\vec{e}_a \tag{3}$$

$$= \sum_a \frac{\exp(2\pi iak/N)}{\sqrt{N}}\vec{e}_{a-1} \tag{4}$$

$$= \exp(-2\pi ik/N)\sum_{a-1} \frac{\exp(2\pi i(a-1)k/N)}{\sqrt{N}}\vec{e}_{a-1} \tag{5}$$

$$= \exp(-2\pi ik/N)\vec{v}^k \tag{6}$$

v^k is the eigenvector with eigenvalue $\exp(-2\pi ik/N$. These are the same points in the complex plane as used in the proof of orthogonality.

(j) T has a basis of N eigenvectors with (different) eigenvalues $\lambda_k = \exp(-2\pi ik/N)$. Since all eigenvalues satisfy $\lambda_k^{-1} = \lambda_k^*$, the matrix T is unitary.

(k) Since A and T commute, the eigenvectors of T (plane waves) help finding eigenvectors of A. Here, they are simply the eigenvectors of A. The eigenvalues can be calculated directly as $\langle v^k|A|v^k\rangle$. No further work is needed to diagonalize A. For T_1, for example, $v^0 = 1/\sqrt{2}\begin{pmatrix}1\\1\end{pmatrix}$ and $v^1 = 1/\sqrt{2}\begin{pmatrix}1\\-1\end{pmatrix}$, and it is easy to see that these are indeed eigenvectors of A_1. Both vectors have a uniform "electron density" $|v_a^k|^2 = 1/\sqrt{2} = 1/\sqrt{N}$ for all k and a.