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Physica A 316 (2002) 453–468

PHYSICA A

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Stability of money: phase transitions in an Ising economy

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Received 20 June 2002

Abstract

The stability of money value is an important requisite for a functioning economy, yet it critically depends on the actions of participants in the market themselves. Here we model the value of money as a dynamical variable that results from trading between agents. The basic trading scenario can be recast into an Ising-type spin model and is studied on the hierarchical network structure of a Cayley tree. We solve this model analytically and observe a phase transition between a one-state phase, always allowing for a stable money value, and a two-state phase, where an unstable (inflationary) phase occurs. The onset of inflation is discontinuous and follows a first-order phase transition. The stable phase provides a parameter region where money value is robust and can be stabilized without fine tuning.

PACS: 89.65.Gh; 05.50.+q; 64.60.Cn

Keywords: Econophysics; Dynamics of money; Spin model of markets; Ising model; Cayley tree; Phase transition

1. Introduction

One of the astonishing facts in economics is the widely observed stability of the value of money [1]. In models of economic activity often well-defined mechanisms are introduced that ensure this property, e.g. by assuming a central agent or market maker who supervises the global dynamics and enforces market clearance. However, real markets already function solely on the basis of the interactions between trading agents [2], raising interesting questions about the validity of equilibrium approaches based on

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PII: S0378-4371(02)01218-9

central agents [3]. This includes the question how the basic dynamics of a decentralized economy can lead to the emergence of money without any explicit central processes of fixing global variables [4,5]. Simple numerical models of exchange of goods between agents [6,7] as well as experimental studies [8,9] support the scenario that the value of money appears as a dynamical variable that results from the dynamics of trading itself.

Money as a free parameter in a system of trading agents has been studied by Bak et al. [10] recently, who cast the problem into a picture consisting of simple agents and flows of money and goods between them. They place the agents on a line, s.t. each trader sells goods to his left neighbor and buys products from his right neighbor. Combining this system with a periodic boundary condition by closing the line to a circle, they observe that the value of money, in general, converges to a stable state and emerges as a dynamical phenomenon in this setting. They conclude that the general picture of this model will also apply to the more complicated heterogeneous networks of agents that in general dominate economy.

However, as the dynamics of this model crucially depend on a very specific choice of the boundary condition, and as a higher dimensional scenario as well as hierarchies between traders may fundamentally change the dynamics, we would like to complement this model by a spatial trading model, offering an alternative interpretation of Jevons' motivation to understand the emergence of money [11]. We will study trading on a hierarchical network which allows us to include the interesting aspect of hierarchy in the monetary business. Also, moving to higher dimensions bears the interesting possibility that a trader with more than two neighbors has extra degrees of freedom to optimize himself by choosing appropriate deals and partners. Finally, we will reformulate this model in terms of an Ising type spin model that can be solved explicitly.

In the next section we will introduce the basic trading model on a network with dimension greater than one. Section 3 is devoted to the problem of competing agents in the presence of a variable money value. In Section 4, we solve an Ising spin realization of the model and study its phase transitions and the conditions for a stable value of money.

2. A network trading model

Let us consider a model where an agent N sells goods which are traded via $N-1$ intermediary agents to consumers at level $n=0$. This is called the selling mode. The goods are returned by a second chain where agent N buys goods, which are traded via $N-1$ different intermediaries from $n=0$ (buying mode). Combining both, buying and selling chain, one obtains the circular geometry of Ref. [10]. Let us now allow the more general scenario that in the selling (or buying) mode each agent can sell to (buy from) $z-1$ agents. The linear chains ($z=2$) are replaced by the so-called Cayley tree with z neighbors (see Fig. 1). The agents are located at the sites or nodes, while goods and money flow along the links of the tree. The agents (n,i) are indexed by the distance n from the right-hand side of the tree. The index i distinguishes different agents at the same distance and will be written only if necessary. For

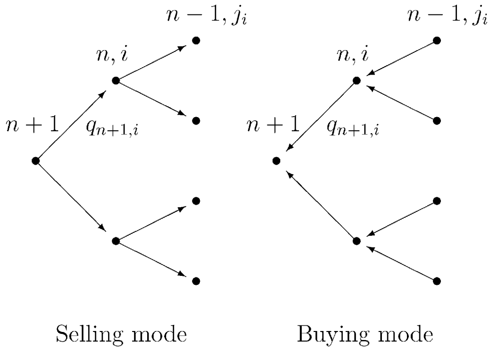

Fig. 1. Selling mode and buying mode for $z=3$ on a Cayley tree. The arrows show the flows of goods $q_{n+1,i}$, while money flow $g_{n+1,i}$ is opposite to $q_{n+1,i}$ at each link and is not explicitly shown.

the amount of goods $q_{n,i}$ flowing between agents n and $n-1, i$ we use the normalized variable

$$\bar{q}_{n,i} = \frac{1}{(z-1)^{n-1}} q_{n,i} . \tag{1}$$

The amount of traded goods is described by two utility functions. If an agent n sells q at the price p he gains the utility

$$u_n^{(S)} = I_n \bar{q} p - \tilde{c}(\bar{q}) . \tag{2}$$

I_n denotes the value of money and $\tilde{c}(\bar{q})$ the decrease of u by losing q. Similarly, if the agent buys q at price p the utility reads

$$u_n^{(B)} = d(\bar{q}) - I_n \bar{q} p . \tag{3}$$

It is important to use the normalized flow of goods (1) instead of q for the following reason. In the monopolistic equilibrium all the money values I_n are the same and all goods are conserved. Therefore, the goods q_n increase with $(z-1)^{n-1}$. Then, utilities (2) and (3) express the assumption that an agent level of n gets the same utility by trading $q_n = (z-1)q_{n-1}$ as the agents at level $n-1$ trading q_{n-1}. As in Ref. [10] each agent n can choose its own money value I_n, the amount of bought goods q_{n+1}, and the price $p_{n,i}$ for sold goods $q_{n,i}$. For a meaningful problem one has to use the following assumption, also made in Ref. [10]. The time scale on which the q or p change is much shorter than the scale of changing the money values. Therefore we can optimize the coupled system (2) and (3) with fixed values of I_n. An additional dynamics must be used for finding I_n from $q(I)$ and $p(I)$. For the utility functions d and $\tilde{c}$ power laws have been used in Ref. [10]. This property is not really needed. It is sufficient for $\tilde{c}$ to increase faster than, and for d less than linearly for large $\bar{q}$. To avoid algebraic complications, we here use for d a power law

$$d(x) = \frac{1}{\beta} x^{\beta} \quad \text{with } \beta < 1 \tag{4}$$

and for $\tilde{c}$ a power law only in the example of Section 4

$$\tilde{c}(x) = \frac{1}{\alpha} x^{\alpha} \quad \text{with } \alpha > 1 . \tag{5}$$

In the general case, $\tilde{c}$ must have positive first and second derivative. Having performed the optimization all quantities can be expressed by the Legendre transform of $\tilde{c}(s^{1/\beta})$ denoted by $c(r)$. For the power law (5) we get

$$c(r) = \frac{\alpha - \beta}{\alpha\beta} (\beta r)^{\alpha/(\alpha-\beta)} . \tag{6}$$

The optimization is slightly different in the buying or selling mode. In the latter we have for each agent n

$$u_n^{(B)} = \frac{1}{\beta} \bar{q}_{n+1}^{\beta} - I_n \bar{q}_{n+1} p_{n+1} \quad n = 0, \dots, N-1 , \tag{7}$$

$$u_n^{(S)} = I_n \sum_i [\bar{q}_{ni} \;\, p_{ni} - \tilde{c}(\bar{q}_{ni})] \quad n = 1, \dots, N . \tag{8}$$

Since $\sum_i \tilde{c}(\bar{q}_i) < \tilde{c}(\sum_i \bar{q}_i)$ the agents n will handle each selling to agents $n-1, i$ separately, and not lump all requests q_{ni} into a single order. The optimization begins at $n = 0$, where only $u_n^{(B)}$ is present. The maximum of (7) leads to the value $\bar{q}_1$. This value $\bar{q}_{1i}(p_{1i})$ is used to optimize p_{1i} in $u_1^{(S)}$ given by (8). This procedure is repeated to the top agent N. The resulting values of traded goods and money flow $g_{n,i}$ from $n-1, i$ to n

$$g_{n,i} = q_{n,i} p_{n,i} \tag{9}$$

are given by

$$q_{n,i} = (z-1)^{n-1} \left[c' \left(\frac{I_n}{I_{n-1,i}} \right) \right]^{1/\beta} , \tag{10}$$

$$g_{n,i} = (z-1)^{n-1} \frac{1}{I_{n-1,i}} c' \left(\frac{I_n}{I_{n-1,i}} \right) . \tag{11}$$

One sees that the goods flow and the valued money flow $I_{n,i} g_{n,i}$ only depend on the ratios I_n/I_{n-1}, but not on the absolute scale of I. The value of utilities in (7) and (8) at the maximum are given by

$$u_n^{(B)} = \frac{1-\beta}{\beta} c' \left(\frac{I_{n+1}}{I_n} \right) \tag{12}$$

and by

$$u_n^{(S)} = \sum_{i=1}^{z-1} c \left(\frac{I_n}{I_{n-1,i}} \right) . \tag{13}$$

The buying mode can be treated with the same method. It can be obtained from the selling mode by interchanging at each link $(n; n-1, i)$ the adjacent I_n and I_{n-1}.

This leads to

$$q_{n,i} = (z-1)^{n-1}\left[c'\left(\frac{I_{n-1,i}}{I_n}\right)\right]^{1/\beta},$$

$$g_{n,i} = (z-1)^{n-1}\,\frac{1}{I_n}\,c'\left(\frac{I_n - 1, i}{I_n}\right) \tag{14}$$

and the utility functions at maximum

$$u_n^{(B)} = \frac{1-\beta}{\beta}\sum_{i=1}^{z-1} c'\left(\frac{I_{n-1,i}}{I_n}\right),$$

$$u_n^{(S)} = c\left(\frac{I_n}{I_{n+1}}\right). \tag{15}$$

In the case of the linear chain ($z = 2$), the only difference between selling and buying is a reordering of I, which is performed in Ref. [10] by placing the agents on a circle. We can consider the normalized money ratio at site n given by (in the selling mode)

$$\Delta g(n+1, n, i) = (z-1)\frac{g_{n-1,i}}{g_n} = \frac{I_n}{I_{n-1,i}}\left[c'\left(\frac{I_n}{I_{n-1,i}}\right)\Big/ c'\left(\frac{I_{n+1}}{I_n}\right)\right]. \tag{16}$$

Note that Δg is only a function of the ratios

$$r_{n,i} = \frac{I_{n+1}}{I_{n,i}}. \tag{17}$$

Further, money conservation at agent n implies

$$\sum_{i=1}^{z-1}\Delta g(n+1, n, i) = z - 1. \tag{18}$$

For $r > 1$ money is accumulated at agent n, while $r < 1$ means that money has to be borrowed. Therefore in the selling mode $r < 1$ implies an inflation, while values $r > 1$ imply deflation. In the buying mode r is essentially replaced by $1/r$ such that Δg is given by

$$\Delta g(n+1, n, i) = \frac{I_{n+1}}{I_n}\left[c'\left(\frac{I_{n-1,i}}{I_n}\right)\Big/ c'\left(\frac{I_n}{I_{n+1}}\right)\right] \tag{19}$$

and the reversed statements are true.

In the case of $z = 2$ in Ref. [10], the condition of money conservation has been applied. Both, the strategy of storing money (Scrooge McDuck mode), as well as the strategy of spending unlimited amounts of money (Donald Duck mode) are punished. There in each step of the update of I_n the condition

$$\Delta g = 1 \tag{20}$$

is imposed. The change of I_n results in new q, p values and the procedure is repeated until convergence to (20) is reached (the additional delay in Ref. [10] only changes the time scale, but not equilibrium (20)).

For $z > 2$ money conservation involves a sum of Δg over i. To fix the money flow $g_{n,i}$ to agent n, extra conditions are needed. Such a condition may result from the cooperation between agents $n-1, i$ connected to n. Suppose agent n sells the amount $q = \sum_i q_{n,i}$ which is bought by agents $n-1, i$. If they do not cooperate, one agent may choose its $I_{n-1,i}$ such that the sum is exhausted. Then the system will collapse into a linear chain. If they cooperate, they optimize their common utility

$$u^{(B)} = \sum_i \frac{1-\beta}{\beta} \bar{q}_{n,i}^{\beta} \tag{21}$$

as function of $I_{n-1,i}$ subject to the condition $\sum q_{n,i} = q$ since the $q_{n,i}$ are unique functions of the $I_{n-1,i}$. For $\beta < 1$ $u^{(B)}$ has a maximum for equal $q_{n,i}$ which implies $I_{n-1,i}$ is independent of i. Therefore we have condition (20) also valid for $z > 2$. In terms of the ratios r it reads

$$c'(r_n) = r_{n-1} c'(r_{n-1}) . \tag{22}$$

This recursion formula for the ratios r_n exhibits the stable fixed point $r_n = 1$, since both $c'(1)$ and $c''(1)$ are positive. The value r_{N-1} is arbitrary. After a transient region, the r_n for $n \ll N$ are equal 1. For power laws the recursion can be solved explicitly. r_n depends only on the ratio of the exponents

$$\gamma = \frac{\beta}{\alpha} \tag{23}$$

which can be called the relative elasticity of the utility functions, and is given by

$$\log r_n = \gamma^{N-1-n} \log r_{N-1} . \tag{24}$$

The same method can be applied in the buying mode. Now r'_0 can be chosen arbitrarily due to the replacement $r_n \to 1/r_n$:

$$\log r'_n = \gamma^n \log r'_0 . \tag{25}$$

Both (24) and (25) can be used to obtain I_n resp. I'_n for the buying mode. In the selling mode, money is accumulated at agent N and the agents at $n = 0$ have to borrow money. In order to 'recycle' the money, one can connect $n = 0$ and N with a second tree in the buying mode where agents $n = 0$ sell other goods q' over this second tree to agent N. From $I_0 = I'_0$ and $I_N = I'_N$ the constants r_{N-1} and r'_0 can be eliminated with the result

$$I_n = I_0 \left(\frac{I_N}{I_0} \right)^{\gamma^{N-n}} , \tag{26}$$

$$I'_n = I_N \left(\frac{I_0}{I_N} \right)^{\gamma^n} . \tag{27}$$

In both (24) and (25) terms $\gamma^N \ll 1$ have been neglected in the exponent. The money values I_N of agent N and I_0 of the agents at $n = 0$ are free constants. Their choice depends on the relative weight the agents place on the utilities in the buying or selling mode. A seller dominated market leads to $I_N > I_0$. In Fig. 2, we show I_n and I'_n as function of n for $\gamma = 1/4$ and $N = 11$. I_n (I'_n) are constant over a wide range and

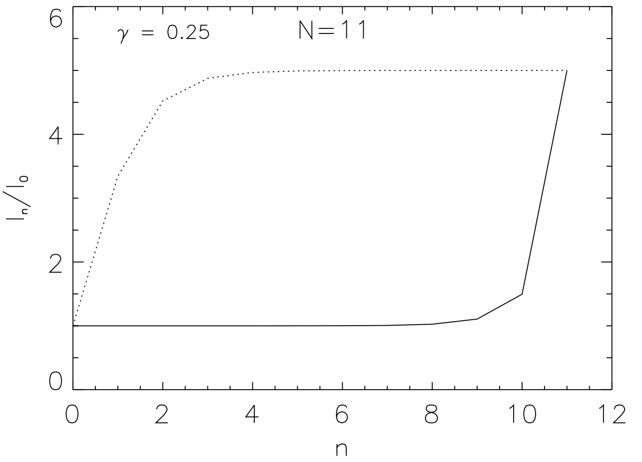

Fig. 2. The money values I_n/I_0 for the selling mode (solid line) and I_n/I_0 for the buying mode (dotted line) as function of n. A seller dominated market with $I_N/I_0 = 5$ has been assumed. The increase of I_n/I_0 near N exhibits the "peanuts effect".

change in the last (first) two steps to the values imposed by the boundary conditions. Constant money values are achieved even when they are different in the selling and buying mode. This shows that the assumption of periodic boundary conditions made in Ref. [10] is crucial for constant money values derived from money conservation $\Delta g = 1$ and not just minimizing the finite size effects as in physical problems.

Another consequence of the recursion is the "peanuts effect": Consider the normalized flows of goods $\bar{q}_n$ in a seller dominated market, using a power-law ansatz for utility functions. They are constant for $n \ll 1$ and increase near $n = N$. The ratio

$$\frac{\bar{q}_N}{\bar{q}_0} = \left(\frac{I_N}{I_0}\right)^{1/(\alpha-\beta)} \tag{28}$$

may take large values, such that also $u_N^{(S)}/u_0^{(B)}$ becomes large. This remarkable feature seems to have induced an unfortunate german banker to publicly call the credits given to small customers at $n = 0$ as "peanuts" (a statement that was not agreed upon by the broad public). [1]

Up to now, the number $z - 1$ of neighbors $n - 1$ adjacent to agent n did not play any rôle given their money value $I_{n-1,i}$ has been chosen equally. In the next chapter we use a dynamics to reach the equilibrium condition (20) from an arbitrary initial state, including thermal noise. The utility function for updating the I_n may have other maxima besides the maximum described by (20). This we investigate in the next section.

[1] Hilmar Kopper, CEO, Deutsche Bank, during a press conference in April 1994, called a loss of 25 million US$ caused by a construction consortium's bankruptcy, "peanuts". This was perceived as being quite cynical, as this loss, while not threatening the bank, indeed seriously hit a large number of small contractors.

3. Utility function for money values

The dynamics of Ref. [10] for the money value I_n is based on the conservation of money flux expressed by $\Delta g = 1$ in the case $z = 2$. This method has several disadvantages. It is completely deterministic and does not allow for noise. More importantly, it does not involve the agents whose utility functions are minimal for the monopolistic equilibrium $r = 1$. Even a possible utility function for the dynamics would be rather complicated, since Δg on a Cayley tree connects agent $n+1$ with agents $n-1, i$ corresponding to a next to next neighbor interaction. In addition, we encounter for $z > 2$ the difficulty that money conservation in Eq. (18) does not determine the dependence of $g_{n,i}$ on i. To improve and to generalize the method of Ref. [10] the dynamics of the money values will be based on an utility function H. Then the noise can be described by a Boltzmann distribution. H is the sum of two parts: One part H_M contains the effect of the money authorities, the second H_A is due to the agents. The latter should involve all agents equally. The simplest choice corresponds to a sum over all utilities $u^S + u^B$. The key observation is that the utilities depend on variables $q_{n,i}$ or $r_{n-1,i} = I_n / I_{n-1,i}$, which are defined on the links $x = (n; n-1, i)$ of the lattice. Moreover, the sum of utilities can be rearranged into a sum over links x

$$H_A = \sum_{agents} u^S + u^B = \sum_x u_A(r_x) \tag{29}$$

with u_A given in the selling mode by

$$u_A(z) = c(z) + \frac{1-\beta}{\beta} c'(z) . \tag{30}$$

The money authority part must favor $\Delta g(n+1, n, i) = 1$. This establishes money conservation and a certain cooperation of the agents n, i to prefer equal money values $I_{n,i}$. Since due to Eq. (16) $\Delta g = 1$ only involves neighboring ratios, this suggests that one should consider the model on the dual lattice which is obtained by replacing the links $(n+1; n)$ of the Cayley tree by nodes x, and the nodes n by $z-1$ dimensional hypertetraeders. This dual lattice for $z = 3$ is called a cactus and is depicted in Fig. 3. $\Delta g(x, y)$ are variables defined on the links x, y of the dual lattice. Nonvanishing values of Δg exist only on the links x, y depicted by the dotted lines in Fig. 3 which are denoted by $x > y$. We model H_M by a sum over all links $x > y$ of a utility function $u_M(\Delta g)$ having a maximum at $\Delta g = 1$. So we arrive at the following utility function for the dynamics of r:

$$H(r) = \sum_x u_A(r_x) + \sum_{x>y} u_M(\Delta g(x, y)) . \tag{31}$$

Possible equilibrium states without noise correspond to the maximum of (31). Thermal noise is introduced by assuming that the equilibrium distribution $w(r)$ for r is Boltzmann distributed with the utility function (31)

$$w(r) \sim e^{\beta_T H(r)} . \tag{32}$$

β_T corresponds to the inverse temperature and $\beta_T \to \infty$ would be the deterministic limit. There exist many dynamics having (32) as equilibrium distribution. Particular

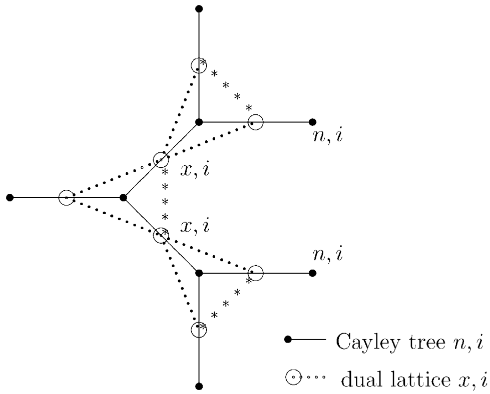

Fig. 3. The Cayley tree and its dual lattice for $z=3$. On the links denoted by $*$ the Δg variables are absent.

interesting are local algorithms as the Glauber [12] dynamics or the Metropolis algorithm [13]. In the latter a randomly chosen agent n selects a new I_n' thereby changing its neighboring ratios r_x to r_x'. The change I_n' is accepted with probability

$$p = \mathrm{e}^{\beta_T \min(0,\Delta H)}\,, \tag{33}$$

where $\Delta H = H(r') - H(r)$ denotes the change in the utility function (31). It only involves the neighboring r_x which are known to the agents by the money- and goods flows at n.

In the following section we discuss a realization of (31) within an Ising-type model. There has been a tradition of using Ising and similar spin models in economic theory [14]. Here, using an Ising formulation has the advantage that probabilities (32) for the average r_x or correlations can be computed explicitly on a Caley tree [15].

4. Phase transitions in the Ising model

In the deterministic limit $\beta_T \to \infty$ the utility function (31) should lead to the state $r_x=1$, corresponding to the absolute maximum of H. However, there may be additional local maxima with $r \neq 1$ which are frozen if the thermal noise vanishes. To study this possibility we consider the following simplified version of (31). We allow only small deviations of r from 1 and parametrize r by a two valued function with one value 1 and the other r_0 close to 1

$$r_x = r_0^{(1+\sigma_x)/2} \tag{34}$$

with an Ising spin variable $\sigma_x = \pm 1$. In addition we assume for the utility function c a power law as in (5). The Boltzmann weight (32) is a product of site factors

$$G^{(0)}(\sigma_x) = \mathrm{e}^{\beta_T u_A(\sigma_x)} \tag{35}$$

and link factors

$$G^{(1)}(\sigma_x, \sigma_y) = \mathrm{e}^{\beta_T u_M(\Delta g)} \tag{36}$$

with Δg derived from (19) for the buying mode and from (16) for the selling mode. In the latter we obtain

$$\Delta g(\sigma_x, \sigma_y) = r_0^{(1-\gamma+\sigma_y-\gamma\sigma_x)/(2(1-\gamma))} . \tag{37}$$

For r_0 close to 1, we can expand u_M around 1 and obtain

$$G^{(1)}(\sigma_x, \sigma_y) = \begin{pmatrix} e^{-K(1-\gamma)^2} & e^{-K\gamma^2} \\ e^{-K} & 1 \end{pmatrix}_{\sigma_x,\sigma_y} \tag{38}$$

with the money conservation constant

$$K = \left(-\frac{\beta_T u_M''(1)}{2}\right)\left(\frac{\alpha \ln r_0}{\alpha - \beta}\right)^2 . \tag{39}$$

In (38) the irrelevant factor $\exp(\beta u_M(1))$ has been omitted. In the same way we obtain for $G^{(0)}$

$$G^{(0)}(\sigma_x) = e^{(z-1)L\delta_{\sigma_x,1}} \tag{40}$$

with the self-interest constant

$$L = \frac{\beta_T}{z-1}\left[u_A(r_0) - u_A(1)\right] . \tag{41}$$

Using (38) and (40) the Boltzmann equilibrium distribution for the dynamical variables σ can be written as

$$w(\sigma) = \frac{1}{Z}\prod_x G^{(0)}(\sigma_x)\prod_{y<x} G^{(1)}(\sigma_x, \sigma_y) . \tag{42}$$

The normalization factor Z follows from the condition $\sum_{\{\sigma\}} w(\sigma) = 1$. The distribution for a single spin $w_1(\sigma_x) = \sum_{\sigma\neq\sigma_x} w(\sigma)$ or two spins $w_2(\sigma_x\sigma_y) = \sum_{\{\sigma\neq\sigma_x,\sigma_y\}} w(\sigma)$ can be calculated recursively [17]. For this purpose we introduce the tree distribution $T_n(\sigma_x)$ of length $|x| = n$ corresponding to the product of all factors $G^{(0)}$ and $G^{(1)}$ on a dual tree starting at x, which is summed over all spins σ_y with $|y| \geqslant |x|$

$$T_n(\sigma_x) = \frac{1}{Z_T}\sum_{\{\sigma_y, y>x\}}\prod_{|y|\geqslant|x|}\left(G^{(0)}(\sigma_y)\prod_{|y'|\geqslant|y|} G^{(1)}(\sigma_y, \sigma_{y'})\right) . \tag{43}$$

Z_T is chosen such that $\sum_\sigma T(\sigma) = 1$. For agent N (43) yields the equilibrium distribution $w_1(\sigma_N)$, for agents $|x| < N$ the tree distribution $T_n(\sigma_x)$ is a conditional probability related to $w_1(\sigma_x)$. According to (43) a tree of length n can be expressed by trees of length $n-1$ in the following way:

$$T_n(\sigma) = G^{(0)}(\sigma)\sum_{\sigma_1,\ldots,\sigma_{z-1}}\prod_{i=1}^{z-1} G^{(1)}(\sigma, \sigma_i)T_{n-1}(\sigma_i) . \tag{44}$$

Any function $T(\sigma)$ depending on a variable $\sigma = \pm 1$ can be parametrized as

$$T_n(\sigma) = a_n(w_n\delta_{\sigma,1} + \delta_{\sigma,-1}) . \tag{45}$$

Carrying out the summation in (44) we find a recursion relation for a_{n+1} and w_{n+1} in terms of a_n and w_n. For the latter this reads

$$w_{n+1} = f(w_n), \tag{46}$$

$$f(w) = \left[e^{-K\gamma^2+L} \, \frac{1+e^{(2\gamma-1)K}w}{1+e^{-K}w} \right]^{z-1} \tag{47}$$

which allows the recursive calculation of w_n if the values w_n at $n=0$ are given. The mean value of r_x for the top agent $x=N$ is related to w_N by the inflation parameter

$$M = \left\langle \frac{\ln r_x}{\ln r_0} \right\rangle = \langle \delta_{\sigma_x,1} \rangle_T = \frac{w_N}{1+w_N} \,. \tag{48}$$

Therefore $w_N \sim 0$ expresses the preference for $r_x = 1$, whereas $w_N \gg 1$ leads to $r_x \sim r_0$. In physical problems $2M-1$ corresponds to the magnetization. Since our utility function is not symmetric under $\sigma \to -\sigma$ the disordered state $M = 1/2$ has no particular meaning. Here only the fully magnetized states are interesting. Inflation parameter $M = 0$ corresponds to the monopolistic state and $M = 1$ implies inflation ($r_0 < 1$) or deflation ($r_0 > 1$). Particularly interesting are stable fixed points w_n of recursion (47). These are solutions independent of n for $n \gg 1$, especially independent of the boundary values w_0. They correspond to a homogeneous value of the inflation parameter on the lattice. If more than one fixed point exists, the system can exhibit different phases. It is a particular property of the Cayley tree that the values at the boundary decide which phase is adopted [16,17]. On a normal finite dimensional lattice only one phase would be thermodynamically stable. The form of $f(w)$ shows that the fixed point equation $w=f(w)$ can have either one or two solutions satisfying the stability condition $|f'(w)| < 1$. Depending on the values of K, L and γ there can be a one-state phase (OSP) with a unique value of M or a two-state phase (TSP) with two possible values. In Fig. 4, we show for the numerical solution of $w=f(w)$ with $z=3$ neighboring agents the inflation parameter $M(w)$ as a function of K for several L values and $\gamma = 0.25$. At low L there will be a unique solution OSP in which M tends to zero for large values K of the money conservation term in H. M increases with the self-interest L of the agents. For sufficiently large L a switch into the TSP with two possible values of M occurs. Still the monopolistic equilibrium can be achieved for large K. The fixed point equation can be only solved numerically, the calculation of the phase boundaries requires solution of quadratic equations. One finds that TSP only occurs if the following two conditions are satisfied. K has to be larger than a critical value given by

$$K_c = \frac{1}{\gamma} \ln \frac{z}{z-2} \tag{49}$$

and L has to be bounded by

$$L_-(K) < L < L_+(K) \,. \tag{50}$$

For the following we need only the asymptotic form of $L_\pm(K)$ for $K \gg 1$

$$L_\pm = K\left(\gamma^2 + \frac{1}{z-1}\right) - 2\gamma K \begin{cases} \frac{1}{z-1}\,, \\ 1\,. \end{cases} \tag{51}$$

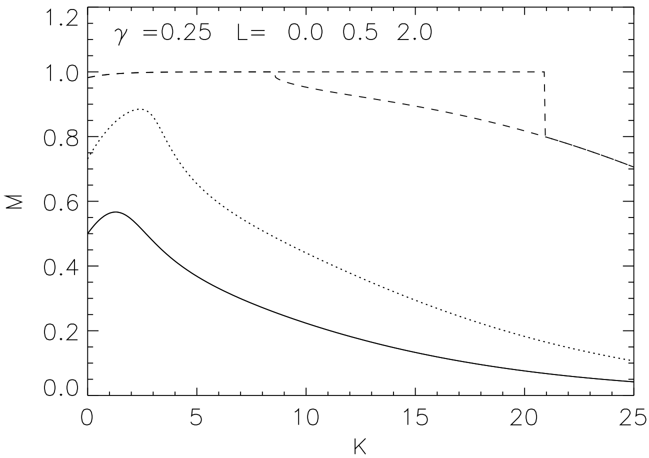

Fig. 4. The inflation parameter M as a function of K for $\gamma = 0.25$ and $z = 3$. The L values are 0 (solid line), 0.5 (dotted line), and 2.0 (dashed line). For $L = 0, 0.5$ the system is in the one state phase. For $L = 2$ and $8.5 < K < 20.9$, the system allows two possible fixed points (two-state phase).

For the linear chain ($z = 2$) discussed in Ref. [10] K_c becomes infinite and a phase transition to TSP cannot occur. The second condition (50) explains why a window for the TSP phase is observed in Fig. 4. Another feature of the model is the dependence on the elasticity γ. For values of $\gamma < \gamma_c$ with

$$\gamma_c = 1 - \sqrt{\frac{z-2}{z-1}}\,, \tag{52}$$

the lower bound L_- is always positive. For fixed L and $K \to \infty$ one always ends up in the OSP in agreement with what we have seen in Fig. 4. Choosing a value $\gamma = 0.6 > \gamma_c$ we show in Fig. 5, the inflation parameter M with $z = 3$ as function of K for various L values. Above K_0 with $L = L_-(K_0) > 0$ the system is always in the TSP with M values near 0 or 1 corresponding to ratios of money values $I_n/I_{n-1} = 1$ or r_0. Even in the deterministic limit $K \to \infty$ the inflationary solution cannot be avoided. The boundaries of the TSP phase in the $(K, L/K)$ plane are shown in Fig. 6 ($\gamma = 0.25 < \gamma_c$) and Fig. 7 ($\gamma = 0.6 > \gamma_c$) for $z = 3$. In Fig. 6, the regions where M is smaller than 0.5 (0.1) are indicated by the dotted (dashed) line, which occur outside of the TSP region. Therefore small values of M are guaranteed in the limit of large K. In contrast for $\gamma > \gamma_c$ the region of small M lies entirely in the TSP region, as seen from Fig. 7. The OSP can be obtained only for negative L which implies $r_0 < 1$ which is against the agents interest in the selling mode. On the other side for $\gamma > \gamma_c$ small values of M can be obtained already at moderate K. The phase transition crossing the bounds $L_\pm$ from OSP to TSP will be in general a first-order transition, since M can change discontinuously by ΔM. If one approaches the end points of TSP near K_c given by (49), the discontinuity vanishes with a power law according

$$\Delta M \sim (K - K_c)^{1/2} \tag{53}$$

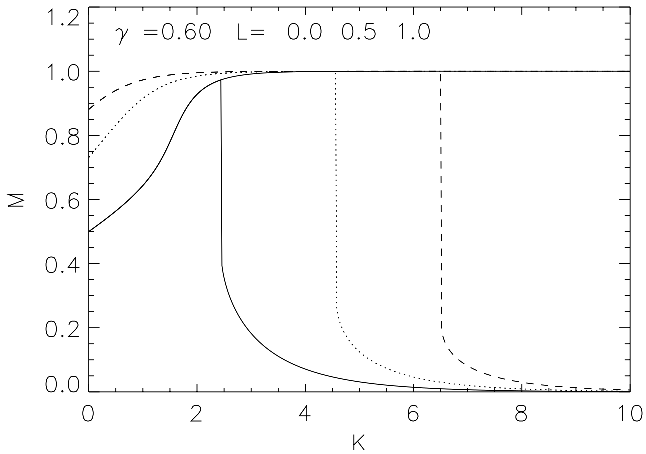

Fig. 5. The inflation parameter M from Eq. (48) as function of K for $\gamma=0.6$ and $z=3$ and various L values. There exists a critical $K(L)$ where the system changes from the OSP into the TSP with one value $M_1 \sim 1$ and one value with $M_0 \sim 0$.

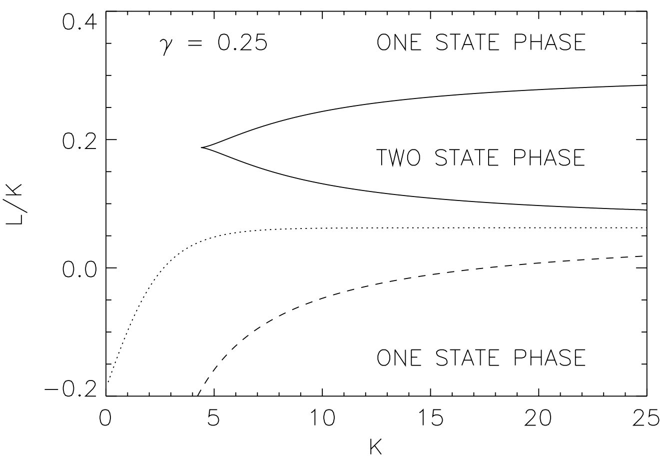

Fig. 6. Phase diagram for $\gamma = 0.25$ and $z = 3$ in the $(L/K, K)$ plane. The solid lines show the critical curves $L_\pm(K)$ where the system changes from the OSP to the TSP ($L_-(K) < L < L_+(K)$). Along the dotted (dashed) line $M = 0.5(0.1)$ holds. Above the dotted (dashed) line $M > 0.5(0.1)$, below $M < 0.5(0.1)$.

indicating a second-order phase transition of the mean field class. With increasing number z of neighboring agents the boundaries of the TSP degenerate into straight lines $L_+ = K\gamma^2$ and $L_- = K\gamma(\gamma - 2)$ implying presence of only the TSP for $L/K < \gamma^2$. The same method can be applied to the buying mode, where agent N buys goods via the tree from agents at $n = 0$. One obtains a similar recursion formula as (47) with

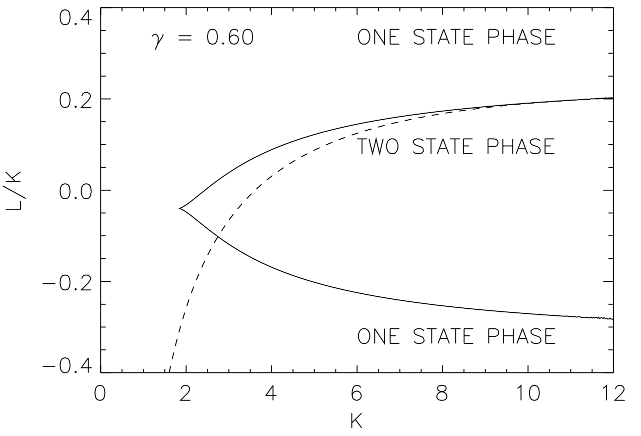

Fig. 7. Phase diagram for $\gamma = 0.6$ and $z = 3$ in the $(K, L/K)$ plane. The solid lines show the critical curves $L_\pm(K)/K$ as in Fig. 6. Along the dashed line one solution for w leads to $M = 0.1$. Below this line M can be smaller than 0.1.

value L', K' and γ' obtained by the replacement

$$L' = -L, \quad K' = \gamma^2 K \quad \text{and} \quad \gamma' = 1/\gamma\,. \tag{54}$$

This leads to qualitatively similar phase transitions.

The money value regulating authorities can achieve a stable economy with an inflation parameter $M = 0$ for given agent parameters L and γ by the choice of large K. The success depends on the value of the elasticity ratio $\gamma = \beta/\alpha$. Very different utilities $\tilde{c}(q)$ and $d(q)$ lead to $\gamma \ll 1$. In this case the system remains in the OSP and the desired result is obtained for large K. For similar utilities $\tilde{c}(q)$ and $d(q)$ we expect $\gamma \sim 1$ and TSP occurs with $M_0 \sim 0$ and $M_1 \sim 1$. Which solution is obtained depends on the boundary values of the agents at $n = 0$. Since their utility function (12) increases with r_0 they prefer a value $r_0 > 1$ leading to increasing money values from $n = 0$ to N (deflation). Additional measures as indirect taxes are required to persuade the $n = 0$ agents to choose the solution M_0. Alternatively one can close the selling tree by a second tree in the buying mode, where the agents at $n = 0$ sell their goods (e.g. labor) through a tree to the top agents. In general, such a mechanism should exist in order to recycle the money flow from $n = 0$ to N in the selling mode. In this case an inflationary value $r_0 < 1$ is preferred. Combining both trees indeed allows the intermediate state $r_0 = 1$ to be reached, as desired in a stable economy.

5. Conclusions

In this article we considered a trading model of agents on the hierarchical network of a Cayley tree, treating money values as dynamical variables. The claim of Ref. [10] that constant money values should result independently of geometry and utility functions of the agents does not appear to be entirely true. Even in the case of a linear chain, imposing money conservation at each agent we find constant I, however, different in the selling and buying mode leading to the "peanuts effect". Only within the periodic boundary conditions of Ref. [10] these constants are the same.

When agents are allowed to choose between neighbors, as for $z > 2$, additional dynamical phenomena may occur, dependent on whether agents cooperate or not. We include this as an optimization problem between nearest neighbors and next to nearest neighbors which, moving the model to the dual lattice (the cactus in $z = 3$) still can be described in terms of nearest neighbor interactions (now between links). An elegant simplification of this model in terms of an Ising model allows to include noise and to explicitly solve the model. The phases of this Ising version of the model correspond to different dynamical regimes of the economy. The main result is the existence of a TSP above a critical money conservation parameter $K_c = (1/\gamma)\ln z/(z-2)$ with critical curves separating the OSP from the TSP. In the TSP one observes a first-order phase transition between an inflationary phase and a phase with stable money value. Whether such a phase transition can occur, depends on the exponents (or elasticities) of the utility functions for buying or selling only. For very different functions the elasticity parameter $\gamma = \beta/\alpha$ will be small and the system can remain in a OSP with stable money value at $K \to \infty$. If one increases γ beyond a critical value $\gamma_c = 1 - \sqrt{(z-2/(z-1)}$ the TSP is inevitable. If the utility functions are similar ($\gamma \approx 1$) the inflationary phase can occur also in the limit $K \to \infty$. In contrast to the linear model ($z = 2$), the equilibrium properties depend on both, the utility functions (γ) and the geometry (z).

These findings are obtained by approximating the ratios of money values I_{n+1}/I_n by discrete Ising variables with only two values and, less important, using power laws for the utilities. The main motivation for these approximations is the possibility to carry out most calculations analytically. The assumption of power laws seems to be not too restrictive. The first assumption of two valued variables can be relaxed by using a larger number of q different values for I_{n+1}/I_n. The resulting q-state Potts model with only nearest neighbor interactions, for $K > 0$ has a similar phase structure as the Ising model [17]. The chaotic behavior observed in this model [18] can, not entirely surprising, occur only for negative K where the authorities aim for inflation. Therefore, the Ising models with ferromagnetic coupling ($K > 0$) should be representative for the general case.

Acknowledgements

The authors thank Thomas Lux and Dietrich Stauffer for helpful discussions and comments.

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