Forces in the double pendulum

For the engineering of mechanical systems with a complex interplay of regular and chaotic behavior it is important to know the forces involved. It is shown how they can be computed and their time development evaluated. Characteristic features of periodic, quasiperiodic, and chaotic motion are identified. Classical methods such as Fourier transform and various statistics are used and compared to a redundant version of wavelet analysis. The latter is proposed as the most informative coherent representation of the distribution of times, frequencies, and amplitudes.

Keywords: double pendulum, chaos, forces of constraint, wavelets

1 Introduction

The planar double pendulum has served as a paradigm for chaotic dynamics in classical mechanics, ever since it was discovered that it displays one of the most beautiful examples of the transition to global chaos via the decay of a last surviving Kolmogorov-Arnold-Moser (KAM) torus [8, 9]. The character of its motion changes dramatically as the energy is increased from zero to infinity. At low energies, it represents a typical case of coupled harmonic oscillators, hence in this limit it is integrable. At very high energies, it is again integrable because the total angular momentum is a second conserved quantity besides energy. At intermediate energies, however, it exhibits a bewildering richness of more or less chaotic features. Given that the general set of physical double pendulums is a three-parameter family of systems, it should not be surprising that only part of the full complexity has been revealed to date. Attempts to do so employed Melnikov’s method [3] and extensive numerical studies [10].

In all studies of chaotic dynamics so far, an important aspect has been left out: the analysis of forces and their time development. The present paper is intended to fill this gap. As a practical application for double or multiple pendulums we think of amusement rides. Engineers who plan to build such machines must be in the position to calculate forces and momenta acting on the bearings. We show how they are calculated and discuss methods for studying their long time behavior. The analysis is easily extended from two to more degrees of freedom.

The basic tools are of course well known. We use Lagrange’s equations of the first kind to identify the forces of constraint and to calculate their time course from the kinematics of the angles. This is the subject of Section 2. It turns out that to evaluate the forces in the general case, two more parameters \( \mu_1 \) and \( \mu_2 \) must be considered besides the three characteristics \( A, \alpha, \beta \) for the kinematics. But rather than making any attempt to explore this five dimensional parameter space, we illustrate the principles with the standard choice (22).

In Section 3 we briefly recall the main features of the kinematic behavior in its dependence of the total energy. We then turn to the forces and in Section 4 derive limits for their maximum possible values. This may suffice as a first estimate of their order of magnitude, but ignores all the interesting details of their time development. The following two sections are the main body of our paper. In Section 5 we discuss standard methods of evaluation in relation to periodic, quasiperiodic, and chaotic modes of behavior. Time series are displayed in different representations, Fourier spectra, autocorrelation functions and various histograms are determined. It is argued that while these methods are well suited to describe periodic and quasiperiodic behavior, they are less apt to cope with chaos. Therefore we advocate, in Section 6, the use of wavelet transforms. They lead to a pictorial representation which we think is an intuitively appealing and adequate medium to evaluate the characteristics of motion with “time dependent power spectra”.

2 Basic equations

Consider two rigid bodies 1 and 2 with centers of gravity \( S_1 \) and \( S_2 \), the first of which is allowed to rotate about a fixed horizontal axis \( A_1 \), the second coupled to the first along an axis \( A_2 \) which is fixed in the first body, see Fig. 1. \( A_2 \) is parallel to \( A_1 \). The configuration of this planar double pendulum is described by the two angles \( \varphi_1 \) and \( \varphi_2 \) as shown in the figure.

Let \( m_1 \) and \( m_2 \) be the masses of the two pendulums, \( \Theta_1^s \) and \( \Theta_2^s \) their moments of inertia with respect to their respective centers of mass. We denote the distances between \( A_i \) and \( S_i \) by \( s_i \), and by \( a \) the distance between...
the two axes. In order that \((\varphi_1, \varphi_2) = (0, 0)\) be the stable equilibrium position, we must require
\[
s_2 \geq 0 \quad \text{and} \quad m_1 s_1 + m_2 a \geq 0. \quad (1)
\]
Within that constraint, \(s_1\) or \(a\) may be negative. There are three unstable equilibria \((\varphi_1, \varphi_2) = (0, \pi), (\pi, 0),\) and \((\pi, \pi)\).

We shall be interested in the motion of this planar double pendulum and in the forces acting on the axes, along the directions \(A_1 S_1\) and \(A_2 S_2\). The standard procedure to derive equations of motion and expressions for the forces of constraint, is the method of Lagrange’s equations of the first kind, i.e., we start with the assumption that \(S_1 = (x_1, y_1)\) and \(S_2 = (x_2, y_2)\) can move freely in the \((x, y)\) plane, and later use Lagrange parameters to eliminate two degrees of freedom by means of the constraints
\[
f_1(r_1) := r_1 - s_1 = 0 \quad \text{and} \quad f_2(r_2) := r_2 - s_2 = 0. \quad (2)
\]
Here \(r_1\) and \(r_2\) are polar coordinates defined by
\[
(x_1, y_1) = (r_1 \sin \varphi_1, -r_1 \cos \varphi_1),
(x_2, y_2) = (r_1, y_1)
+ ((a - s_1) \sin \varphi_1 + r_2 \sin \varphi_2, -(a - s_1) \cos \varphi_1 - r_2 \cos \varphi_2). \quad (3)
\]

It is straightforward to write down the kinetic and potential energy contributions \(T_i\) and \(V_i\) of the two pendulums:
\[
T_1 = \frac{1}{2} (\Theta_1^2 + m_1 r_1^2) \dot{\varphi}_1^2 + \frac{1}{2} m_1 r_1^2 \quad (4)
\]
\[
T_2 = \frac{1}{2} (\Theta_2^2 + m_2 r_2^2) \dot{\varphi}_2^2 + \frac{1}{2} m_2 \left[ (a + r_1 - s_1)^2 \dot{\varphi}_1^2 + 2(a + r_1 - s_1) r_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_2 - \varphi_1) + (a + r_1 - s_1) r_2 \dot{\varphi}_1 \dot{\varphi}_2 \sin(\varphi_2 - \varphi_1) + r_1^2 + r_2^2 + 2 r_1 r_2 \cos(\varphi_2 - \varphi_1) \right] \quad (5)
\]
\[
V_1 = m_1 g (s_1 - r_1 \cos \varphi_1) \quad (6)
\]
\[
V_2 = m_2 g (a - (a + r_1 - s_1) \cos \varphi_1) + m_2 g (s_2 - r_2 \cos \varphi_2) \quad (7)
\]

From the Lagrangian \(L = T_1 + T_2 - V_1 - V_2\) and Hamilton’s principle
\[
\delta \int (L + \lambda_1 f_1 + \lambda_2 f_2) dt = 0 \quad (8)
\]
we recover the constraints by variation of \(\lambda_i\). This implies \(\dot{r}_1 = 0\) and \(\dot{r}_2 = 0\) which simplifies the rest of the equations. Variation of \(r_1\) produces the forces of constraint. The force \(\lambda_1\) acts on axis \(A_1\) in the direction of \(A_2\),
\[
\lambda_1 = (m_1 + m_2) g \cos \varphi_1 + (m_1 s_1 + m_2 a) \dot{\varphi}_1^2
+ m_2 s_2 \left[ \dot{\varphi}_2^2 \cos(\varphi_2 - \varphi_1) + \dot{\varphi}_2 \sin(\varphi_2 - \varphi_1) \right], \quad (9)
\]

Figure 1: Planar double pendulum. Axis \(A_1\) is fixed in space, \(A_2\) fixed in the first body. \(A_1, A_2,\) and the center of gravity \(S_1\) are assumed to lie in the same plane which forms the angle \(\varphi_1\) with the direction of gravity. The configuration of the outer pendulum is given by \(\varphi_2\).
and \( \lambda_2 \) acts on \( A_2 \) in the direction of \( S_2 \):
\[
\lambda_2 = m_2 g \cos \varphi_2 + m_2 s_2 \ddot{\varphi}_2^2 \\
+ m_2 a \left( \dot{\varphi}_1^2 \cos(\varphi_2 - \varphi_1) - \dot{\varphi}_1 \sin(\varphi_2 - \varphi_1) \right).
\]
(10)

To evaluate the time development of these forces, we need the equations of motion for the angles \( \varphi_i \). They derive from the Lagrangian with \( r_i = s_i = \text{const} \):
\[
L = \frac{1}{2} (\Theta_1 + m_2 a^2) \dot{\varphi}_1^2 + \frac{1}{2} \Theta_2 \dot{\varphi}_2^2 + m_2 s_2 a \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_2 - \varphi_1) \\
- (m_1 s_1 + m_2 a) g (1 - \cos \varphi_1) - m_2 s_2 g (1 - \cos \varphi_2).
\]
(11)

Here the moments of inertia are related to the respective points of suspension, \( \Theta_1 = \Theta_1^s + m_1 s_1^2 \). It is convenient to choose appropriate units in order to reduce the number of parameters in the equations. With lengths measured in units of \( a \), times in units of \( \sqrt{a/g} \), energies in units of \( (g/a) \Theta_2 \), the scaled Lagrangian becomes
\[
L = \frac{1}{2} A \dot{\varphi}_1^2 + \frac{1}{2} \dot{\varphi}_2^2 + \alpha \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_2 - \varphi_1) - \beta (1 - \cos \varphi_1) - \alpha (1 - \cos \varphi_2),
\]
(12)

with only three independent dimensionless parameters
\[
A = \frac{\Theta_1 + m_2 a^2}{\Theta_2}, \quad \alpha = \frac{m_2 s_2 a}{\Theta_2}, \quad \beta = \frac{(m_1 s_1 + m_2 a)}{\Theta_2}.
\]
(13)

From the positivity of the moments of inertia, or of the kinetic energy, one has the requirement \( A > \alpha^2 \). Introducing canonical momenta by \( p_i = \partial L/\partial \dot{\varphi}_i \), the system's Hamiltonian becomes
\[
H = \frac{1}{2} \sum p_i \dot{T}_{ij} p_j + \alpha \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_2 - \varphi_1) + \beta (1 - \cos \varphi_1) + \alpha (1 - \cos \varphi_2).
\]
(14)

It is of course a constant of motion,
\[
H = E = \text{const},
\]
(15)

and in general there is no other. From the kinetic energy \( T = \frac{1}{2} \sum p_i T_{ij} p_j \), we identify a matrix \( T = T_{ij} \)
\[
T = \frac{1}{A - \alpha^2 \cos^2(\varphi_2 - \varphi_1)} \left( \begin{array}{cc}
1 & -\alpha \cos(\varphi_2 - \varphi_1) \\
-\alpha \cos(\varphi_2 - \varphi_1) & A
\end{array} \right).
\]
(16)

Using this matrix, we find that the equations of motion may be written in the form
\[
\begin{pmatrix}
\ddot{\varphi}_1 \\
\ddot{\varphi}_2
\end{pmatrix} = T \begin{pmatrix}
-\beta \sin \varphi_1 + \alpha \varphi_2^2 \sin(\varphi_2 - \varphi_1) \\
-\alpha \sin \varphi_2 - \alpha \dot{\varphi}_1^2 \sin(\varphi_2 - \varphi_1)
\end{pmatrix}.
\]
(17)

The scaling of forces in units of \( (g/a^2) \Theta_2 \) gives
\[
\lambda_1 = (\mu_1 + \mu_2) \cos \varphi_1 + \beta \dot{\varphi}_1^2 + \alpha \left( \varphi_2^2 \cos(\varphi_2 - \varphi_1) + \dot{\varphi}_2 \sin(\varphi_2 - \varphi_1) \right)
\]
(18)

\[
\lambda_2 = \mu_2 \cos \varphi_2 + \alpha \dot{\varphi}_2^2 + \mu_2 \left( \dot{\varphi}_1^2 \cos(\varphi_2 - \varphi_1) - \ddot{\varphi}_1 \sin(\varphi_2 - \varphi_1) \right)
\]
(19)

where \( \mu_1 \) and \( \mu_2 \) are two additional parameters,
\[
\mu_1 = \frac{m_1 a^2}{\Theta_2}, \quad \mu_2 = \frac{m_2 a^2}{\Theta_2}.
\]
(20)

Eliminating \( \ddot{\varphi}_1 \) and \( \ddot{\varphi}_2 \) with the help of (17), we obtain the forces of constraint in the form
\[
\lambda_i = \lambda_i(\varphi_1, \varphi_2, \varphi_1^2, \varphi_2^2).
\]
(21)

These equations describe the motion and forces of constraints for all possible double pendulums. The standard case with two equal mass points, at the ends of massless rods of equal length, has the values
\[
A = 2, \quad \alpha = 1, \quad \beta = 2, \quad \mu_1 = \mu_2 = 1.
\]
(22)

All numerical computations in this paper will be done with this particular set.
3 Characteristics of the motion

The equations of motion derived from the Lagrangian (12) contain a surprisingly rich dynamical behavior [8, 9]. The four-dimensional phase space is foliated by energy surfaces $H = E$ whose topological type changes at the equilibrium values $E = 0$, $2\alpha$, $2\beta$, and $2(\alpha + \beta)$. The energy surface is a 3-sphere $S^3$ for energies between 0 and $\min(2\alpha, 2\beta)$, and a 3-torus $T^3$ for energies above $2(\alpha + \beta)$.

At low energies $E \to 0$, the motion is a superposition of harmonic oscillators of eigenfrequencies $\omega_1$ and $\omega_2$ given by

$$\omega_{1,2}^2 = \frac{1}{2} \left[ \frac{1}{4} - \frac{1}{\alpha^2} \right] \left( \frac{\beta}{\alpha} + \frac{\alpha A}{\beta} \pm \sqrt{\left( \frac{\beta}{\alpha} - 1 \right)^2 + 4\beta^2} \right).$$

With the standard parameters this is

$$\omega_{1,2}^2 = 2 \pm \sqrt{2} \quad \Rightarrow \quad \omega_1 = 1.848, \quad \omega_2 = 0.765.$$ (24)

The frequency ratio is $W = \omega_1/\omega_2 = 1 + \sqrt{2}$. Such a motion is of course integrable because the energies of the individual eigenmodes are conserved quantities. The energy surface $S^3$ is foliated by invariant tori $T^2$.

The motion is also integrable at high energies $E \to \infty$, or vanishing gravity, which may be realized by turning the double pendulum into a horizontal plane. The second constant of motion besides the total energy is then the total angular momentum $L = p_1 + p_2$. The possible values of $L$ are $-L_{\text{max}} \leq L \leq L_{\text{max}}$ where

$$L_{\text{max}}^2 = 2E(A + 1 + 2\alpha).$$ (25)

A given energy surface is again foliated by invariant Liouville-Arnold tori, but as discussed in [8], these are of different type depending on whether $L^2$ is smaller or larger than $L_{\text{sep}}^2$, with

$$L_{\text{sep}}^2 = 2E(A + 1 - 2\alpha).$$ (26)

For large total angular momenta, $L^2 > L_{\text{sep}}^2$, each given $L$ corresponds to a torus where $\varphi_1$ proceeds in a rotational motion (the direction depending on the sign of $L$), and the relative angle $\phi = \varphi_2 - \varphi_1$ oscillates around the value 0. Physically speaking, the outer pendulum performs oscillations in a centrifugal potential. At the separatrix values $L = \pm L_{\text{sep}}$, the relative angle $\phi$ reaches the position $\pi$ which corresponds to an unstable rotation of the folded double pendulum. At lower values, $L^2 < L_{\text{sep}}^2$, both angles $\varphi_1$ and $\phi$ rotate. For a given value of $L$, there exist two tori of this kind, differing in the sense of rotation.

At intermediate energies, the motion is non-integrable. An extensive discussion of the onset of chaos, as $E$ is lowered from infinity towards zero, was given in the movie [9]. Around $E \approx 10$, the system provides a beautiful example of the onset of global chaos via the decay of the “golden torus”, i.e., of the last surviving KAM torus whose winding number is the golden section.

A convenient representation of this behavior is given in terms of Poincaré sections and maps. The equations of motion are integrated, and whenever the double pendulum is in the stretched configuration $\varphi_1 = \varphi_2$, or $\phi = 0$, with $\phi > 0$, we record the values $(\varphi_1, L)$ of the two-dimensional “surface of section” in the three-dimensional energy surface. The set of values $(\varphi_1, L)$ is called the Poincaré plane $\mathcal{P}$, and the map $P: \mathcal{P} \to \mathcal{P}$ which assigns a given intersection $(\varphi_1, L)$ the next one along the trajectory, is called the Poincaré map.

A comprehensive survey is given in the series of nine pictures in Fig. 2.

Fig. 2a shows the case $E = \infty$. The horizontal lines are intersections of the invariant tori $L = \text{const}$ with the Poincaré surface of section. They are each generated from 500 successive iterations of the Poincaré map. The colors indicate the type of motion. Yellow: positive rotation of $\varphi_1$ and oscillation of $\phi$; orange: negative rotation of $\varphi_1$ and oscillation of $\phi$; blue: both $\varphi_1$ and $\phi$ oscillate in positive direction; green: negative rotation of $\varphi_1$ and positive rotation of $\phi$. (The time reversed motion of blue and green type is not represented in this picture; it could be obtained with the condition $\phi = 0, \dot{\phi} < 0$.)

The white lines are images of the line $\varphi_1 = 0$ under one iteration of $P$. At given $L$, their value $\Delta \varphi_1$ tells how much the angle $\varphi_1$ advances for each period of the relative angle $\phi$. If $\Delta \varphi_1/2\pi$ is a rational value $m/n$, with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, then the motion is periodic: $m$ periods of $\varphi_1$ correspond to $n$ periods of $\phi$. At the boundary between blue and green tori there is a conspicuous case of $\Delta \varphi_1 = 0$ which develops into a strong resonance at finite values of $E$, cf. Fig. 2b: under the influence of gravity, the first pendulum oscillates while the second rotates with the same period.

The sequence of pictures in Fig. 2 shows that two major chaotic regions develop from the boundaries between yellow/blue (upper chaos) and orange/green (lower chaos). Rational tori with low integers $m, n$ develop into
resonances whereas irrational tori survive until about $E = 10$. The last torus to survive before the two chaotic regions merge at $E = 10.352\ldots$ [6], can be shown to possess the golden winding ratio $\Delta \varphi_1/2\pi = (\sqrt{5} - 1)/2$. Around $E = 4$, all stable resonances have disappeared; as far as numerical analysis can tell, the motion seems to be ergodic. When $E$ decreases further, new resonances emerge from the chaos, and below $E \approx 1$, invariant tori dominate the picture again.

4 Limits to the forces of constraint

Before we study the time development of the forces of constraint $\lambda_i$, we shall determine the maximum values $\lambda_i^{\text{max}}$, for given values of $\varphi_1$, in the stretched configuration $\varphi_1 = \varphi_2$. To this end, we consider the energy

$$E = \frac{1}{2}A \dot{\varphi}_1^2 + \frac{1}{2} \varphi_1^2 + \alpha \dot{\varphi}_1 \dot{\varphi}_2 + (\alpha + \beta)(1 - \cos \varphi_1)$$

(27)

and the force $\lambda_1$ as given by (18):

$$\lambda_1 = (\mu_1 + \mu_2) \cos \varphi_1 + \beta \varphi_1^2 + \alpha \varphi_2^2.$$  

(28)
The constancy of energy, dE = 0, and the extremum condition dλ₁ = 0 define two ellipses in the (ϕ₁, ϕ₂) plane. It is elementary to find that these ellipses are tangent to each other on the lines

\[ \frac{\dot{\varphi}_2}{\dot{\varphi}_1} = \frac{1}{2\alpha^2} \left( \beta - \alpha A \pm \sqrt{(\beta - \alpha A)^2 + 4\beta\alpha^3} \right) =: \gamma_{1,2}. \] (29)

It turns out that the plus sign leads to the minimum of λ₁ at the conditions assumed, whereas the minus sign gives the maximum we are looking for. Hence we take γ₂ = γ in the following formulas.

Inserting this into the energy equation (27), we obtain

\[ \dot{\varphi}_1^2 = \frac{2(E - (\alpha + \beta)(1 - \cos \varphi_1))}{A + 2\alpha \gamma_{1,2} + \gamma_{1,2}^2}, \] (30)

from which λ₁ max derives as

\[ \lambda_{1\text{ max}} = (\mu_1 + \mu_2) \cos \varphi_1 + 2(E - (\alpha + \beta)(1 - \cos \varphi_1)) \frac{\beta + \alpha \gamma^2}{A + 2\alpha \gamma + \gamma^2}. \] (31)

In complete analogy, the maximum values of λ₂ are obtained as

\[ \lambda_{2\text{ max}} = \mu_2 \cos \varphi_1 + 2(E - (\alpha + \beta)(1 - \cos \varphi_1)) \frac{\mu_2 + \alpha \delta^2}{A + 2\alpha \delta + \delta^2}, \] (32)

where

\[ \delta := \frac{1}{2\alpha^2} \left( \mu_2 - \alpha A - \sqrt{(\mu_2 - \alpha A)^2 + 4\mu_2\alpha^3} \right). \] (33)

It is of course assumed that \( E \leq (\alpha + \beta)(1 - \cos \varphi_1) \); otherwise the angle \( \varphi_1 \) cannot be reached by lack of energy.

Let us interpret these results. At energy \( E = (\alpha + \beta)(1 - \cos \varphi_1) \), only the forces of gravity act on the axes. For larger energies, the extra centrifugal forces increase linearly with \( E \). The rate of increase depends on the relative motion of the two pendulums. It is interesting that at fixed \( E \), the maximum values of both \( \lambda_1 \) and \( \lambda_2 \) occur when the two pendulums move in opposite directions, \( \dot{\varphi}_2/\dot{\varphi}_1 < 0 \). For the standard pendulum (22), we find

\[ \lambda_{1\text{ max}} = 2 \cos \varphi_1 + 2(2 + \sqrt{2})(E - 3(1 - \cos \varphi_1)), \]
\[ \lambda_{2\text{ max}} = \cos \varphi_1 + (3 + \sqrt{5})(E - 3(1 - \cos \varphi_1)). \] (34)

It is not necessarily true that \( \lambda_{1\text{ max}} > \lambda_{2\text{ max}} \). If \( \mu_2 > \beta \), the force on the second axis increases faster than that on the first, hence for sufficiently large energies we then have \( \lambda_{2\text{ max}} > \lambda_{1\text{ max}} \).
Looking at the $\varphi_1$-dependence of these forces, it is obvious that the maximum values are obtained with $\varphi_1 = 0$, when gravity and centrifugal forces act in the same direction. In the neighborhood of the unstable equilibrium $\varphi_1 = \pi$, it is not necessarily true that the maximum forces occur in the stretched configuration $\varphi_1 = \varphi_2$. But similar calculations may be carried out for other cases, such as $(\varphi_1, \varphi_2) = (\pi, 0)$, for example. However, for an estimate as to the range in which the forces may vary, the above considerations are sufficient. In Fig. 3 we give an example for the polar diagrams obtained if we plot the time course of forces and angles, taking $(\lambda_1, \lambda_2, \varphi_1, \varphi_2)$ as polar coordinates. For a typical chaotic trajectory with energy $E = 20$, the actual values of the forces are seen to stay well within the boundaries given by Equations 34. But it is also apparent that if $\lambda_1(t)$ are considered as functions of time, most relative maxima do not even come close to $\lambda_{i}^{max}$. To get a more detailed understanding, we therefore turn our attention to the temporal behavior of the forces.

Figure 4: Time development of the forces $\lambda_1$ (left) and $\lambda_2$ (right) for a periodic trajectory: $E = 50, L = 6.036471$.

Figure 5: Forces $\lambda_1(t)$ (left) and $\lambda_2(t)$ (right) for a quasiperiodic trajectory: $E = 50, L = 1.25$.

Figure 6: Forces $\lambda_1(t)$ (left) and $\lambda_2(t)$ (right) for a chaotic trajectory: $E = 50, L = 10$. 
5 Analysis of the time development

Let us consider typical cases of double pendulum motion. Solving the equations of motion (17) by a standard numerical method, we determine the forces with equations (18) and (19). The question then arises as to how the data are best analyzed.

In the data presented in the following, initial conditions of trajectories were chosen from the symmetry line \( \phi_1 = 0 \) of the Poincaré surfaces of section in Fig. 2. This means that both pendulums start from the hanging position, \( \varphi_1(0) = \varphi_2(0) = 0 \). The initial momenta are determined from \( E \) and \( L \) (taking into account \( \dot{\phi} > 0 \)).

The most complete information is given in terms of plots time \( t \) versus \( \lambda_i(t) \). In Figs. 4 to Figs. 6 this is done for three characteristically different trajectories of energy \( E = 50 \). Choosing initial conditions close to the major resonance, compare Fig. 2b, we obtain the simple picture of Fig. 4. Quasiperiodic motion is shown in Fig. 5, and chaotic motion in Fig. 6. The qualitative difference of the three cases is obvious, but time series of this kind have the problem that they contain too much information. The distinction of periodic, quasiperiodic, and chaotic motion calls for observation over long times, but then the time in itself is not very interesting.

The polar diagrams defined in the previous section discard time as a variable and instead exhibit information on the angle \( \varphi_1 \). The compactness of angular variables allows for visual evaluation over longer times (50 units in all diagrams), and for an intuitively clear-cut distinction of the three types of motion, see Figs. 7 through 9.

Figure 7: Polar diagrams \((\lambda_1(t), \varphi_1(t))\) (left) and \((\lambda_2(t), \varphi_2(t))\) (right) for the periodic trajectory \( E = 50, L = 6.036471 \).

Figure 8: \((\lambda_1(t), \varphi_1(t))\) (left) and \((\lambda_2(t), \varphi_2(t))\) (right) for the quasiperiodic trajectory \( E = 50, L = 1.25 \).

The pictures suggest two kinds of statistical evaluation. From a practical point of view it is interesting to know the distribution \( \rho(\lambda_m) \) of the sizes \( \lambda_m \) of relative maxima of the forces, and the distribution \( \rho(\Delta t_m) \) of time intervals \( \Delta t_m \) between successive maxima. These are important characteristics for the wear on the material. In Figs. 10 and 11 they are plotted for \( \lambda_1 \). The time series used for these calculations were 100 units long. This is
certainly not enough for accurate results, but it suffices to exhibit the differences of the three types of motion. (To produce the histograms, the respective total ranges of $\lambda_m$ and the $\Delta t_m$ interval between 0 and 2 were divided into 50 bins. The absolute values of $\rho$ are reduced to a total of 10 time units.)

The simple periodic motion assumes always the same maximum, with only one value of $\Delta t_m$. Some spread is noticeable in the quasiperiodic case, but not nearly as much as for chaotic motion. In the latter case the force maxima seem to fall into two broad regions, and the time intervals between them vary considerably.

The standard method to extract typical frequencies and times is of course Fourier analysis [1]. Given $\lambda(t)$, the Fourier transform is defined as

$$\hat{\lambda}(\omega) = \int_{-\infty}^{\infty} \lambda(t)e^{-i\omega t} \, dt,$$  

(35)
The Fourier transform contains the same information as the original time series (which may be recovered by inverse transformation), but the power spectrum ignores its phases. The tradeoff is that the frequencies and their relative weights are more clearly borne out. The same information, but with an emphasis on correlation times, is contained in the autocorrelation function

\[ C(\tau) = \int_{-\infty}^{\infty} \lambda(t+\tau)\lambda(t) \, dt = \int_{-\infty}^{\infty} P(\omega) e^{i\omega \tau} \frac{d\omega}{2\pi}, \]

where the last equation is the Wiener-Khinchin theorem.

Figs. 12 and 13 show, respectively, the power spectra \( P(\omega) \) on a logarithmic scale, and autocorrelation functions \( C(\tau) \) normalized to \( C(0) = 1 \), for the same orbits as before. They were determined with the recipes for discrete fast Fourier transforms (FFT) described in [7]. We did not aim for accuracy, but purposefully applied the standard procedure on a relatively short time segment of some 20 units (4096 points with discretization step \( \delta t = 0.005 \)), without adjusting its length to the inherent periods. Despite this moderate computational effort, the different types of motion can easily be distinguished.

In the periodic case, the spectrum ideally consists of a basic frequency and its higher harmonics. The broadening of individual peaks is an artifact of the analysis but does not spoil the essential features of the picture. The strong non-sinusoidal nature of the motion implies large contributions of the higher harmonics. The autocorrelation function ought to come out periodic; its slow decay in Fig. 13 (left) is again a consequence of the discretization and the finite sampling time. The same is true for the quasiperiodic case where the exact spectrum would still be discrete, but with combinations of incommensurate frequencies. The extra peaks are clearly discernible on the artificial background noise. For chaotic motion, both the power spectrum and the autocorrelation function are dramatically different. The spectrum is continuous, and \( C(\tau) \) exhibits a fast decay towards irregular oscillations of low amplitude.

The same analysis may be applied to each trajectory of the double pendulum. The overall distinction of periodic, quasiperiodic, and chaotic character is readily reflected in the results. However, certain finer details of
Figure 14: Time signal of the force $\lambda_1$ (left), its power spectrum (middle) and autocorrelation function (right) for the chaotic trajectory with $E = 6, L = 0$.

chaotic motion are not so easily borne out. Consider, for example, the case $E = 6, L = 0$, i.e., the trajectory starting at the very center of Fig. 2. A typical sample of the time development of the force $\lambda_1(t)$ is shown in the left part of Fig. 14. Every now and then, but in a completely irregular manner, the behavior changes between chaotic and almost quasiperiodic character. In the histograms for maxima and their time differences, see Fig. 15, this results in a fairly broad distribution (which keeps changing with increasing sampling time), but it is hard to detect in this integral picture the characteristic pattern of the motion. This holds even more true for the power spectrum and the autocorrelation function which are given in the middle and right part of Fig. 14: the traces of quasiperiodicity are lost.

Figure 15: Relative frequency of maxima of the forces $\lambda_1(t)$ (left) and relative frequency of time intervals between successive maxima (right) for the chaotic trajectory with $E = 6, L = 0$.

As a last example, consider the case $E = 9, L = 0$. The last KAM torus which is still visible in Fig. 2c, has decayed into a Cantorus and does no longer separate the regions of positive and negative total angular momentum $L$. Its remnants, however, are still an obstacle in phase space so that it takes time for a trajectory to find its way across. In Figs. 16 to 18 we show how the first crossover takes place after some 1500 time units. Up to about time 1480, the motion is genuinely chaotic, with positive $L$, and for a long time after time 1530, it is again chaotic, but with negative $L$, see Fig. 16. During the intermediate 50 time units, the recording of angles $\varphi_1(t)$ and $\varphi_2(t)$ in Fig. 17 exhibits nearly quasiperiodic behavior, reminiscent of the no longer existing golden KAM torus. The time development of the forces, see Fig. 18 is less clear in this respect. The analysis in terms of histograms, see Fig. 19, or power spectrum and autocorrelation function, see Fig. 20, does not reveal any of these features.

For this reason, we now turn to a different procedure, the method of wavelet transforms, which analyzes the system simultaneously in time and frequency domains.
Figure 16: Time development of the total angular momentum $L$ for the trajectory with $E = 9, L = 0$.

Figure 17: Time development of the angles $\phi_1$ (left) and $\phi_2$ (right) for the trajectory with $E = 9, L = 0$.

Figure 18: Time development of the forces $\lambda_1$ (left) and $\lambda_2$ (right) for the trajectory with $E = 9, L = 0$. 
Wavelet transforms are a relatively new method of time series analysis, providing a rather comprehensive and intuitively appealing picture in both time and frequency. They are integral transforms related to windowed Fourier transforms, with considerable freedom in the choice of basis functions. This allows for adaptation to the given problem and for optimization of various aspects, including the speed of computation.

In a windowed Fourier transform, the signal is masked with a time window of given size. This implies that high frequency signals are investigated in finer detail than those in the low frequency domain. For signals with low and high frequency parts the disadvantage is that the different ranges of the spectrum are given unequal attention. Wavelet transforms solve this problem by means of variable time-frequency windows; the signal \( \lambda(t) \) is analyzed with a two-parameter family of basis functions,

\[
\hat{\lambda}(a, b) := \int_{-\infty}^{\infty} \lambda(t) \overline{\psi_{a,b}(t)} \, dt, \tag{38}
\]

where

\[
\psi_{a,b}(t) := |a|^{-\frac{1}{2}} \psi \left( \frac{t - b}{a} \right), \tag{39}
\]

\( a, b \in \mathbb{R}, a > 0 \) [2]. The function \( \psi(t) \in L^2(\mathbb{R}) \) is called a wavelet, provided its Fourier transform \( \hat{\psi}(\omega) \) satisfies the
admissibility condition

\[ C_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty; \]  

(40)

\( C_\psi \) is called the Calderon constant. Besides the usual fast decay required for square-integrability, the condition requires the Fourier transform to vanish at zero.

The value \( b \) is a localization parameter; it determines where in time the signal is analyzed. The value \( a \) is a scaling parameter; it determines the wave length and temporal extent of the wavelet. Its inverse gives the frequency scale; if \( |\hat{\psi}(\omega)| \) has its maximum at \( \omega = \omega_0 \), then \( \psi_{a,b}(t) \) analyzes for contributions around \( \omega_0/a \).

An example of a wavelet \( \psi(t) \) is the Mexican hat function

\[ \psi_M(t) := \frac{2}{\sqrt{3}} \pi^{-\frac{1}{4}} (1-t^2)e^{-t^2} \]  

(41)

with Calderon constant \( C_\psi = \frac{8}{3} \sqrt{\pi} \), see e.g. [2]. This wavelet and its Fourier transform are shown in Fig. 21. Wavelength and total width of the wavelet are the same order of magnitude. The width of the main peak of \( \psi_M(t) \), as given by the zeros, is 2, and the maximum of \( |\hat{\psi}_M(\omega)| \) occurs at \( \omega_0 = \sqrt{2} \).

In the following we take \( \psi(t) = \psi_M(4t) \) as our basic wavelet; the motivation for the rescaling of time comes from the observation that the shortest periods to be resolved in our physical time series \( \lambda_1(t) \) are in the range of 0.5 time units, see Fig. 4. The maximum of \( |\hat{\psi}(\omega)| \) is then at \( \omega_0 = 4\sqrt{2} \).

For the purpose of numerical computation, both wavelet parameters \( a, b \) must be discretized. In a well known version of wavelet analysis, called multiscale analysis (MSA), linear discretization is applied to time \( t \) whereas scaling and localization parameters \( a \) and \( b \) are discretized logarithmically, i.e., they are taken at multiples \( 2^m \), \( m = 0, 1, \ldots \), of some basic values \( a_0, b_0 \). This corresponds to a frequency bisection in each step. The transformation so defined can be viewed as a decomposition of the given signal with respect to a basis in function space, and may be inverted. The theory of MSA can be found, e.g., in [2].

We propose instead to use a scheme that produces redundant information and thereby allows us to see how the various frequency scales are connected. We follow the recipes described in [5] where a number of applications to signal and noise filtering have been discussed.

The discretization \((t, a, b) \rightarrow (n, j, k)\) is performed with

\[ t_n := n\delta t, \quad b_k := kb_0, \quad a_j := ja_0; \quad n, k \in \mathbb{Z}, \quad j = 1, 2, \ldots, j_{\text{max}}. \]  

(42)

We choose \( \delta t = b_0 = 0.02 \) and \( a_0 = 2\delta t \), \( j_{\text{max}} = 500 \). As it is convenient to plot \( j \), or the frequency \( \omega \), on a logarithmic scale, our actual choice of values \( j \) is \( j = \exp(m\varepsilon) \) with \( m = 0, 1, \ldots, m_{\text{max}} = 500 \) and \( m_{\text{max}}\varepsilon = \log j_{\text{max}} \). With the usual sampling procedure for signals we put

\[ \psi_{j,k}(n) := \frac{1}{\sqrt{ja_0}} \psi \left( \frac{n - kb_0}{ja_0} \right) \]  

(43)
and

\[ \hat{\lambda}(j, k) := \sum_n \lambda(n) \tilde{\psi}_{j,k}(n). \]  

(44)

The translation from \( j \) into frequencies \( \omega \) is done with \( \omega = \omega_0/j a_0 = 100 \sqrt{2}/j \).

Consider now the series of Figs. 24 through 27. They are two-dimensional representations where time runs linearly from left to right, \( k = 0, \ldots, 4096 \), corresponding to \( 0 \leq t \leq 82 \), and frequency decreases logarithmically from top to bottom, \( j = 1, \ldots, 500 \), corresponding to \( 141 \geq \omega \geq 0.28 \). The original time series \( \lambda_1(t) \) is shown at the top of each figure. The colors code for the absolute values of \( \lambda_1(j, k) \), dark blue for \( |\lambda_1(j, k)| = 0 \), and red for the maximum value in each case, as shown on the color bar at the bottom of each figure. The numbers below the bars give the correspondence of colors and \( |\lambda_1(j, k)| \).

Let us get used to the representation with the three types of signals \( \lambda_1(t) \) at energy \( E = 50 \), by comparison with Figs. 4 to 6 and the Fourier transforms in Fig. 12. The periodic case with \( L = 6.036471 \) is shown in the upper picture of Fig. 24. The main feature of the picture is the concentration of wavelet power \( |\hat{\lambda}_1(j, k)| \) in a time-independent frequency band, with \( j \) values between 10 and 20, or frequencies \( \omega \) around the peak at 11.3, corresponding to the period of the oscillations (0.55 time units). There is no power on scales with \( j \) larger than about 40. The somewhat conspicuous repetitive pattern in time is not to be given too much attention. It is a consequence of interference between the given time signal and the wavelet as it is shifted along the time axis. Even with purely harmonic signals \( \lambda(t) \), the values of \( \hat{\lambda}(j, k) \) would alternate between positive and negative values, depending on the relative position of the signal’s and the wavelet’s minima and maxima. This interference pattern may be viewed as a fingerprint of the signal’s shape, in analogy to the pattern of higher harmonics in the Fourier power spectrum. In this particular case, we observe the 10:9 resonance between the typical time 0.5 of the basic wavelet and the signal’s period of 0.55.

Similar considerations explain the regular temporal pattern in the \( j \) range between 40 and 100 of the middle part of Fig. 24, where the quasiperiodic case \( L = 1.25 \) is analyzed. The wavelet power is seen to be distributed between two time-independent ranges. The major contribution is approximately the same as in the periodic case, with a slightly higher peak frequency of 12.2, corresponding to a period of about 0.50 time units; the other frequency range is about a factor of 5 lower and displays a strong interference pattern. The structure at the left and right boundary of the bottom of the figure is an artifact of the analysis and should be ignored.

The frequency resolution of the method is limited by the Heisenberg uncertainty relation \( \Delta t \cdot \Delta \omega \geq 1 \) (which holds with the equal sign in the case of the Mexican hat). For the basic wavelet we have both \( \Delta t \approx 1 \) and \( \Delta \omega \approx 1 \), and as a consequence of scaling the relative resolution \( \Delta \omega/\omega \approx 1 \) is everywhere the same. This allows us to determine, for each time \( k \), the dominant frequency contributions to the signal. To do so quantitatively, we consider vertical cuts through the pictures. Fig. 22 shows two examples, one for periodic motion (left), the other for quasi-periodic motion (right). These cuts may be interpreted as time dependent “wavelet power” spectra. (It must however be kept in mind that due to scaling, the true amplitude of oscillation with frequency \( \omega \) is obtained only after multiplying the graphs of Fig. 22 with \( \sqrt{\omega} \).) To eliminate spurious contributions from the interference of wavelet shape and signal, it is advised to average over one unit of time; thus the spectra shown here are averages.
of the 50 $k$ values between times 40 and 41. The graphs should be compared to the corresponding Fourier power spectra in Fig. 12. Rather than analyzing for harmonic contributions (which are no natural feature of the given nonlinear oscillations anyway), the wavelet spectra give us a rough but clear idea of the frequencies contributing to the signal at a given time. Shape and relative width of the individual peaks are everywhere given by the Fourier transform $\psi(\omega)$ of the basic wavelet, see Fig. 21. There is just one peak around $\omega \approx 11$ in the periodic case, and two around the main frequencies in the quasiperiodic case.

The appeal of the method becomes apparent when we now turn to the chaotic case $L = 10$, see the bottom part of Fig. 24. It is obvious that the two main frequency bands are severely distorted by processes with longer time scales. Still we may identify characteristic frequencies at any given time, but these are no longer constant. Three major frequency ranges may be discerned, two of them approximately the same as in the quasiperiodic case. The third and most irregular is the low frequency range. Its structure is made apparent by the blue lines of zero amplitude $\lambda_1(j, k)$ which extend into the high frequency ranges. Neither vertical nor horizontal sections alone can reveal this interplay of scales; the wavelet analysis seems to be the adequate method to display this complex behavior. The picture integrates all information that was collected before. Fig. 6 (left) showed two kinds of maxima of $\lambda_1$: the narrow peaks of high amplitude 300 appear here as the first frequency band from the top, the broader peaks of amplitude around 130 contribute to the second band. The distribution of maxima of a given height can be read off in more detail than in Fig. 10 (right): looking for the corresponding color we may determine where they occur. The same is true for typical recurrence times for peaks of given width (and height).

It may be said that Fig. 24 (bottom) displays weak chaos in that the quasiperiodic character of the motion is only mildly disturbed. As the energy $E$ is lowered, the picture tends to become more chaotic, as Fig. 25 indicates, $E = 20$, $L = 6.5$. The top frequency band is still visible albeit at somewhat lower frequencies due to the smaller energy. But the second band has virtually disappeared while the low frequency irregularities extend further up than before.

At energy $E = 10$, see Fig. 26, the basic frequency band is further lowered, and it appears that some quasiperiodicity with frequencies around $j = 30$ and $j = 100$ can be identified. But again this is strongly interrupted by irregular features of very low frequency.

Fig. 27 (top) presents the wavelet analysis of the case $E = 9$, $L = 0$, that was discussed in Figs. 16 to 20 of the previous section. To see the transition from one chaotic regime to another, the time range needs to be extended here to 8192 points, or 164 units; it spans the same interval as in Fig. 16. The transient regularity with two main frequency bands and very little power in the low frequencies is quite impressive. The Fourier power spectrum in Fig. 20 is much less informative because it integrates over the entire time range and thereby hides the interesting aspects. It is more adequate to describe the system in terms of time dependent wavelet power spectra, two of which are shown in Fig. 23: one for the almost quasiperiodic phase ($1506 < t < 1507$), the other for chaotic motion ($1564 < t < 1565$). Two clear peaks can be distinguished during the regular transition; Fig. 16 shows that they gradually shift towards lower frequencies. Then, in the chaotic phase, three to four peaks appear with changing positions and relative strengths.

An even more dramatic change from fairly regular to chaotic motion is shown in the middle part of Fig. 27, for $E = 8$ and $L = 0.76$. The initial condition is chosen just outside the island of stability around the 1:1 resonance.
in Fig. 2d. For a long time, the motion stays quasiperiodic, with no indication of an impending change. Then suddenly chaos sets in. We think that the wavelet analysis provides the most intuitive pictures of such behavior.

Finally, the picture at the bottom of Fig. 27 represents the strongly chaotic case $E = 6, L = 0$, compare Figs. 14 and 15 of the previous section. The picture is reminiscent of turbulence in the way large structures (low frequencies) decay into smaller and smaller. The broad distribution of maxima and their recurrence times is well borne out.
Figure 24: Wavelet transforms of trajectories with $E = 50$: top: $L = 6.036471$ (periodic), middle: $L = 1.25$ (quasiperiodic), bottom: $L = 10$ (chaotic).
Figure 25: Wavelet transform of the chaotic trajectory with $E = 20, L = 6.5$.

Figure 26: Wavelet transform of the chaotic trajectory with $E = 10, L = 0.9$. 
Figure 27: Wavelet transforms of three chaotic trajectories: top: $E = 9$, $L = 0$, middle: $E = 8$, $L = 0.76$, bottom: $E = 6$, $L = 0$. 
7 Conclusion

Chaos is a long time phenomenon. To infer its existence from the observation of time series, in strict mathematical terms, it is necessary to have an infinitely long sample. Only then is it possible to determine with certainty that the Fourier spectrum is continuous, at least one Lyapunov exponent positive, or - in Hamiltonian systems with \( f \) degrees of freedom - that the stable invariant subsets of \( 2f \) dimensional phase space are of higher dimension than \( f \). When the equations of motion are known, as in the case of double or multiple pendulums, the existence of chaos may also be asserted by analytic means if it is possible to find and characterize transverse homoclinic or heteroclinic orbits. But these are matters of principle, not very relevant to practical applications.

So far chaotic dynamics has not seen many applications in mechanical engineering, despite the fact that “almost all” Hamiltonian systems exhibit chaos, integrability being the exception rather than the rule. In recent years, however, the situation seems to be changing because our understanding of chaos has improved, and its handling become possible due to the availability of fast computers and of algorithms for “chaos control”. Examples of growing interest are the design of amusement rides, and of cranes for reloading ships at sea. In such cases, it is important to know not only the kinematics of the motion but also the concomitant forces; and the relevant time scales are not infinitely long but given by the characteristic times of irregular behavior – typically a few periods of elementary rotations or oscillations.

We chose the example of a double pendulum to demonstrate how these problems might be attacked. The equations of motion of this simple physical system have been known for centuries, but only recently has their rich dynamical complexity been explored. The method of Poincaré sections is an excellent tool to find out how the phase space is partitioned into regular and chaotic regions, but neither does it tell us the temporal behavior of a trajectory nor does it contain information on the forces involved. We used it primarily to look for interesting initial conditions in order to generate time series for angles and forces.

The analysis of such time series is an independent problem and may be performed without knowledge of how the series was obtained. In fact, once the series are given, it does not matter whether they describe angles or forces. As far as angles are concerned, the behavior of double and triple pendulums was analyzed before in [4], with similar methods and results as those described in Section 5. The shortcoming of those classical methods is that by averaging over very long times they are blind for the interesting qualitative features on intermediate time scales, the irregular transitions between more or less well defined types of motion.

We think that a wavelet analysis is the best available method to date for obtaining a comprehensive picture of the distribution of typical times, frequencies, and amplitudes in a chaotic time series. It gives an adequate representation of the turbulence contained in the signal, where by turbulence we mean the hierarchical interplay of longer and shorter time scales, and the observation that power spectra keep changing in time. Averaging them over very long times as in a standard Fourier analysis misses the point that their change is the phenomenon of interest.

It may still be a long way from there to an evaluation of the effects on material wear, perhaps the major concern of a designer of machines. But we strongly suggest to use the wavelet transforms as a starting point from which all relevant information can be deduced.

References

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